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Thermodynamic relations of the Hermitian matrix ensembles

Yang Chen† and Mourad E H Ismail‡
Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2BZ, UK

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Abstract. Applying the Coulomb fluid approach to the Hermitian random matrix ensembles, *universal* derivatives of the free energy for a system of N logarithmically repelling classical particles under the influence of an external confining potential are derived. It is shown that the elements of the Jacobi matrix associated with the three-term recurrence relation for a system of orthogonal polynomials can be expressed in terms of these derivatives and therefore give an interpretation of the recurrence coefficients as thermodynamic susceptibilities. This provides an algorithm for the computation of the asymptotic recurrence coefficients for a given weight function.

We also show that a pair of quasilinear partial differential equations, obtained in the continuum limit of the Toda lattice, can be integrated exactly in terms of certain auxilliary functions related to the initial data, and in our formulation in terms of integrals of the logarithm of the weight function. To demonstrate this procedure we give some examples where the initial data increases along the half line.

Combining identities of the theory of orthogonal polynomials and certain Coulomb fluid relations, a second-order ordinary differential equation (with coefficients determined by the Coulomb fluid density) satisfied by the polynomials is derived. We use this to prove some conjectures put forward in previous papers. We show that, if the confining potential is convex, then near the edges of the spectrum of the Jabcobi matrix, orthogonal polynomials of large degree is uniformly asymptotic to Airy function.

1. Preliminaries

The joint probability distribution denoted by $p(x_1,\ldots,x_N)$, of the eigenvalues, $\{x_j : 1 \leqslant j \leqslant N\}$, for an ensemble of complex $N\times N$ Hermitian random matrices is given by the classical formula, see Weyl [29] and also [24],

$$p(x_1,x_2,\ldots,x_N)\prod_{j=1}^{N}\mathrm{d}x_j := [Z_N]^{-1}\exp[-\Phi(x_1,\ldots,x_N)]\prod_{j=1}^{N}\mathrm{d}x_j \tag{1.1}$$

where

$$Z_N := \left(\prod_{j=1}^{N}\int_K \mathrm{d}x_j\right)\exp[-\Phi(x_1,\ldots,x_N)] \tag{1.2}$$

is the normalization constant, also known as the partition function. Here,

$$\Phi(x_1,\ldots,x_N) := \sum_{1\leqslant j\leqslant N} u(x_j) - 2\sum_{1\leqslant j<k\leqslant N}\ln|x_j-x_k| \tag{1.3}$$

† E-mail address: y.chen@ic.ac.uk
‡ E-mail address: ismail@math.usf.edu

from the point of statistical mechanics, is the total energy of a system of N logarithmically repelling particles obeying Boltzmann statistics subject to a common external potential $u(x)$ in one dimension, and K is a subset of $\mathbb{R}$. For sufficiently large N, we expect this collection of particles can be approximated by a continuous fluid [12] where techniques of macroscopic physics such as thermodynamics and electrostatics can be applied. It has been found that the continuum approach is very accurate and effective, see for example [4–6]. The Coulomb fluid approximation is described by an equlibrium density $\sigma(x)$ which is obtained by minimizing the free-energy functional, $F[\sigma]$,

$$F[\sigma] = \int_J \mathrm{d}x[u(x) + tx]\sigma(x) - \int_J \mathrm{d}x \int_J \mathrm{d}y\sigma(x) \ln|x - y|\sigma(y) \tag{1.4}$$

subject to

$$\int_J \mathrm{d}x\sigma(x) = N \tag{1.5}$$

where the extra potential $tx, (t \geqslant 0)$ is introduced to generate a one-time flow and J is a subset of $\mathbb{R}$. It turns out that with the introduction of the 'time' parameter, t, the thermodynamic relations; which are the central results of this paper, can be easily derived.

Upon minimization, the density $\sigma(x)$ is found to satisfy the integral equation,

$$A = u(x) + tx - 2\int_J \mathrm{d}y\sigma(y) \ln|x - y| \qquad x \in J \tag{1.6}$$

where A is the Lagrange multiplier for the constraint (1.5) and is recognized as the chemical potential for the fluid. Note that A is a constant independent of x for $x \in J$ but both A and σ depend on t and N. In the framework of potential theory, the constant $\frac{A}{2N}$, is known as the Robin constant for the external field $\frac{u(x)}{2N}$, see [26].

The integral equation, (1.6), is converted into an equivalent singular integral equation by taking a derivative with respect to x,

$$u'(x) + t = 2P\int_J \frac{\mathrm{d}y}{x - y}\sigma(y) \qquad x \in J. \tag{1.7}$$

At this stage J is taken to be a subset of $\mathbb{R}$. For the purpose of this work we shall assume that $u(x)$ is convex for $x \in \mathbb{R}$. It follows that $u''(x) \geqslant 0$ almost everywhere. We shall not consider the case where $u''(x) = 0$ almost everywhere, so we assume $u''(x) \geqslant 0$ on a set of positive measure. With this condition on $u(x)$ it follows that J is a single interval denoted by (a, b). Intuitively, this can be understood by using an analogy from elasticity theory [25], where the fluid density $\sigma(x)$ is identified with the pressure under a stamp pressing vertically downwards against an elastic half-plane. If the applied force is moderate, the end points of the interval, a and b, are the points for which the elastic material comes into contact with the rigid stamp. On the other hand if the force applied to the stamp is too great the end points will be fixed as the end points of the boundary of the stamp, if $u(x)$ has compact support. As the potential, $u(x)$, which we are investigating, is supported in $\mathbb{R}$, only a finite amount of force could be applied; the points of contact are determined by two subsidiary conditions to be stated below. There is a further analogy which will help with the determination of the interval J. The potential $u(x)$ can be viewed as the cross section of a container into which a charged fluid (with each molecule carrying the same positive charge) is poured. For a fluid of a fixed amount of N molecules, the fluid density will terminate at the end points of an interval, which are a and b for the case under consideration, to minimize the free energy. This requires that the density vanishes at the end points of the interval. It transpires that the end points a and b satisfy two functional equations to be stated later.

However, if the potential, $u(x)$, is nonconvex, J could be the union of several disjoint intervals. This will be dealt with in a separate paper. In section 2, using the Coulomb fluid method we express the recurrence coefficients as derivatives of the free energy. In section 3, we show how the initial value problem for the continuum Toda lattice can be integrated using the Coulomb fluid method. In section 4, explicit solutions of the Toda lattice and the continuum limit of the Toda equations are obtained. It is found that the time evolution of certain classes of orthogonal polynomials can be obtained explicitly. In section 5, certain identities that involve the time evolution of the zeros of the orthogonal polynomials are derived. In section 6, the differential equation for the single-interval case is derived. This paper concludes with section 7.

We seek a solution of (1.7) which is nonnegative on (a, b). If imposing the boundary conditions $\sigma(a) = \sigma(b) = 0$ lead to a σ satisfying $\sigma(x) \geqslant 0$ on (a, b) then according to the standard theory of singular integral equations [16, 25], the solution of (1.7) is

$$\sigma(x) = \frac{\sqrt{(b-x)(x-a)}}{2\pi^2} P \int_a^b \mathrm{d}y \frac{u'(y)+t}{(y-x)\sqrt{(b-y)(y-a)}} \tag{1.8}$$

and a and b must satisfy the constraint (1.5) as well as the supplementary condition,

$$0 = \int_a^b \mathrm{d}x \frac{u'(x)+t}{\sqrt{(b-x)(x-a)}}. \tag{1.9}$$

Using (1.8) the normalization condition becomes

$$N = \frac{1}{2\pi} \int_a^b \mathrm{d}x \frac{x(u'(x)+t)}{\sqrt{(b-x)(x-a)}}. \tag{1.10}$$

The end points of the support of the density, a and b, that are solutions of (1.9) and (1.10) are denoted by $a(N, t)$ and $b(N, t)$. Note that N and t are independent variables.

Sometime the boundary conditions that σ vanishes at the endpoints of J do not lead to a solution $\sigma(.)$ which is nonegative on J. In this case other forms of solutions of (1.7) can be used. We will encounter this situation in section 2.

2. Orthogonal polynomials and the Coulomb fluid

If $\{p_n(x, t)\}_{n\geqslant 0}$ is a system of monic polynomials orthogonal with respect to the weight function

$$w(x, t) := \exp[-(u(x) + tx)] \qquad x \in \mathbb{R} \tag{2.1}$$

and has the orthogonality relation

$$\int_{-\infty}^{\infty} \mathrm{d}x\, w(x, t) p_m(x, t) p_n(x, t) = h_n(t)\delta_{m,n} \tag{2.2}$$

with $h_n(t)$ the square of the L^2 norm, then it follows that $\{p_n(x, t)\}$ satisfies a three-term recurrence relation,

$$x p_n(x, t) = p_{n+1}(x, t) + \alpha_n(t) p_n(x, t) + \beta_n(t) p_{n-1}(x, t) \qquad n \geqslant 0 \tag{2.3}$$

where $\alpha_n, n \geqslant 0$ are real and $\beta_n > 0, n > 0$ with the convention $\beta_0 p_{-1}(x, t) := 0$. The recurrence coefficients $\{\alpha_n\}$ and $\{\beta_n\}$ can be expressed in terms of the h_ns of (2.2) as

$$\beta_n(t) = \frac{h_n(t)}{h_{n-1}(t)} \tag{2.4}$$

$$\alpha_n(t) = -\frac{1}{h_n(t)} \frac{\mathrm{d}h_n(t)}{\mathrm{d}t}. \tag{2.5}$$

Consider the partition function,

$$Z_N(t) = \left(\prod_{1 \leqslant j \leqslant N} \int_{-\infty}^{\infty} \mathrm{d}x_j \right) \exp[-\Phi(x_1, \ldots, x_N, t)] \tag{2.6}$$

where

$$\Phi(x_1, \ldots, x_N, t) := \Phi(x_1, \ldots, x_N) + t \sum_{1 \leqslant j \leqslant N} x_j. \tag{2.7}$$

According a theorem of Dyson [24], the above multiple integral is also related to the h_ns and its value is

$$Z_N(t) = N! \prod_{0 \leqslant j \leqslant N-1} h_j(t). \tag{2.8}$$

Thus, one can find the time dependence of recurrence coefficients once we know the L^2 norms of the polynomials. On the other hand if we define the free energy as

$$\exp[-F_N(t)] := \frac{Z_N(t)}{N!} = \prod_{0 \leqslant j \leqslant N-1} h_j(t) \qquad h_0(t) = 1 \tag{2.9}$$

we can then apply (2.4) and (2.5) and express the recurrence coefficients in terms of the free energy in the form

$$\alpha_N(t) = \frac{\mathrm{d}}{\mathrm{d}t}[F_{N+1}(t) - F_N(t)] \tag{2.10}$$

$$\beta_N(t) = \exp[-(F_{N+1}(t) + F_{N-1}(t) - 2F_N(t))]. \tag{2.11}$$

Now for sufficiently large N, the free energy is approximated by the free energy obtained from the Coulomb fluid technique and the finite difference indicated in (2.10) and (2.11), can be expressed as derivatives of the Coulomb fluid free energy given by equation (1.4) with respect to the particle number N. Thus

$$\alpha_N(t) = \frac{\partial}{\partial t}\left[\frac{\partial F}{\partial N} + \frac{1}{2}\frac{\partial^2 F}{\partial N^2} + \mathrm{O}\left(\frac{\partial^3 F}{\partial N^3}\right)\right] \tag{2.12}$$

$$\beta_N(t) = \exp\left[-\frac{\partial^2 F}{\partial N^2}\right]\left(1 - \frac{1}{12}\frac{\partial^4 F}{\partial N^4} + \mathrm{O}\left(\frac{\partial^6 F}{\partial N^6}\right)\right) \tag{2.13}$$

where in equations (2.12) and (2.13) F denotes the free energy obtained by substituting the equilibrium density into the free-energy functional, (1.4).

We now proceed to the computation of the derivatives of F with respect to N. From thermodynamics

$$\frac{\partial F}{\partial N} = A. \tag{2.14}$$

To mathematically verify this, we compute the partial derivative of F with respect to N and make use of the boundary conditions, $\sigma(a) = 0 = \sigma(b)$. A simple calculation shows,

$$\frac{\partial F}{\partial N} = \int_a^b \mathrm{d}x \frac{\partial\sigma(x)}{\partial N}\left[u(x) + tx - 2\int_a^b \mathrm{d}y \ln|x-y|\sigma(y)\right] = A\int_a^b \mathrm{d}x \frac{\partial\sigma(x)}{\partial N} = A \tag{2.15}$$

since

$$\int_a^b \mathrm{d}x\sigma(x) = N. \tag{2.16}$$

From this we have

$$\frac{\partial A}{\partial t} = x - 2\int_a^b \mathrm{d}y \frac{\partial\sigma(y)}{\partial t} \ln|x-y|. \tag{2.17}$$

Since $\frac{\partial A}{\partial t}$ is constant for $x \in (a, b)$, we find by computing the derivative of (2.17) with respect to x,

$$1 = 2P\int_a^b \frac{\mathrm{d}y}{x-y}\frac{\partial\sigma(y)}{\partial t} \qquad x \in (a, b). \tag{2.18}$$

Using the fact that N and t are independent variables

$$\int_a^b \mathrm{d}x \frac{\partial\sigma(x)}{\partial t} = \frac{\partial N}{\partial t} = 0. \tag{2.19}$$

The general solution for $\frac{\partial\sigma(x)}{\partial t}$ in (2.18) is of this form;

$$\frac{\partial\sigma(x)}{\partial t} = \frac{1}{2\pi}\sqrt{\frac{b-x}{x-a}} + \frac{C}{\sqrt{(b-x)(x-a)}}$$

where the second term is the solution of the homogeneous equation. The constant C is $-\frac{b-a}{4\pi}$, found from (2.19). Thus,

$$\frac{\partial\sigma(x)}{\partial t} = \frac{1}{2\pi}\frac{(a+b)/2 - x}{\sqrt{(b-x)(x-a)}}. \tag{2.20}$$

Substituting (2.20) into (2.17) and with the aid of the integrals,

$$\int_0^1 \frac{\mathrm{d}s}{\pi}\frac{\ln s}{\sqrt{s(1-s)}} = -2\ln 2$$

$$\int_0^1 \frac{\mathrm{d}s}{\pi}\sqrt{\frac{1-s}{s}}\ln s = -\ln 2 - \tfrac{1}{2}$$

we find

$$\frac{\partial A}{\partial t} = \frac{a+b}{2} \tag{2.21}$$

and

$$\alpha_N(t) = \frac{a(N,t) + b(N,t)}{2} + \mathrm{O}\left(\frac{\partial^2 A}{\partial t \partial N}\right). \tag{2.22}$$

Taking the partial derivative of A with respect to N we find,

$$\frac{\partial A}{\partial N} = -2\int_a^b \mathrm{d}y \frac{\partial\sigma(y)}{\partial N}\ln|x-y|. \tag{2.23}$$

Noting that $\frac{\partial A}{\partial N}$ is a constant for $x \in (a, b)$, we find by taking a derivative of (2.23) with respect to x that $\frac{\partial\sigma}{\partial N}$ satisfies the integral equation

$$P\int_a^b \frac{\mathrm{d}y}{y-x}\frac{\partial\sigma(y)}{\partial N} = 0 \tag{2.24}$$

where the unique solution, satisfying $\int_a^b \mathrm{d}x \frac{\sigma(x)}{\partial N} = 1$, is

$$\frac{\partial\sigma(x)}{\partial N} = \frac{1}{\pi\sqrt{(b-x)(x-a)}}. \tag{2.25}$$

Therefore,

$$\frac{\partial^2 F}{\partial N^2} = \frac{\partial A}{\partial N} = -\ln\left[\frac{(b(N,t)-a(N,t))^2}{16}\right] \tag{2.26}$$

and

$$\beta_N(t) = \frac{[b(N,t)-a(N,t)]^2}{16}\left[1+\mathrm{O}\left(\frac{\partial^4 F}{\partial N^4}\right)\right]. \tag{2.27}$$

The derivation of formulae (2.22) and (2.27) relies on the assumptions that

$$b(N+1,t)-b(N,t) = \mathrm{o}(b(N,t)) \qquad \text{and} \qquad a(N+1,t)-\alpha(N,t) = \mathrm{o}(a(N,t)).$$

These imply

$$\alpha_{N+1}(t)-\alpha_N(t) = \mathrm{o}(\alpha_N(t)) \qquad \text{and} \qquad \beta_{N+1}(t)-\beta_N(t) = \mathrm{o}(\beta_N(t)).$$

This holds if the recurrence coefficients have polynomial growth in N as in the case of Freud or Erdös weights [15, 20–22]. Thus, our formulae can be applied to compute the large N behaviour of the recurrence coefficients and extreme zeros of polynomials orthogonal with respect to Freud and Erdös weights [20, 23, 8]. However, in the cases where the recurrence coefficients have exponential growth and the polynomials are orthogonal with respect to weak exponential weight, that is $u(x) = \mathrm{O}((\ln|x|)^m)$, where m is an even positive integer, the error terms in (2.22) and (2.27) must be taken into account. The main results in the section, (2.22) and (2.27) are valid for convex $u(x)$ and $u''(x) > 0$ on a set of positive measure. In section 3, we discuss the relationship between orthogonal polynomials and the Toda lattice. In a continuum limit, a procedure for integrating a nonlinear wave equation is given using the Coulomb fluid method.

3. Toda lattice

From equations (2.20) and (2.25) we uncover a dynamical law that governs the motion of the fluid density which in the context of the recurrence relation is the spectral density of the Jacobi matrix, where x is now interpreted as the spectral variable:

$$\frac{\partial\sigma}{\partial t} = \frac{1}{2}[R-x]\frac{\partial\sigma}{\partial N} \tag{3.1}$$

where

$$R(N,t) := \frac{a+b}{2} \tag{3.2}$$

is the centre of mass coordinates. With the introduction of the difference coordinates,

$$r(N,t) := \frac{b-a}{2} \tag{3.3}$$

the normalization condition, (1.10), and the supplementary condition, (1.9), becomes respectively;

$$\frac{2\pi N}{r} = \int_{-1}^{1}\frac{\mathrm{d}s\,s}{\sqrt{1-s^2}}u'(rs+R) \tag{3.4}$$

$$-\pi t = \int_{-1}^{1}\frac{\mathrm{d}s}{\sqrt{1-s^2}}u'(rs+R). \tag{3.5}$$

By taking partial derivatives with respect to N and t, we find

$$\varepsilon_1 \frac{\partial r}{\partial N} + \varepsilon_0 \frac{\partial R}{\partial N} = 0 \tag{3.6}$$

$$\varepsilon_1 \frac{\partial r}{\partial t} + \varepsilon_0 \frac{\partial R}{\partial t} = -\pi \tag{3.7}$$

$$\left[\varepsilon_2 + \frac{2\pi N}{r^2}\right] \frac{\partial r}{\partial N} + \varepsilon_1 \frac{\partial R}{\partial N} = \frac{2\pi}{r} \tag{3.8}$$

$$\left[\varepsilon_2 + \frac{2\pi N}{r^2}\right] \frac{\partial r}{\partial t} + \varepsilon_1 \frac{\partial R}{\partial t} = 0 \tag{3.9}$$

where

$$\varepsilon_k(r, R) := \int_{-1}^{1} \frac{\mathrm{d}s s^k}{\sqrt{1-s^2}} u''(rs + R) \qquad k \geqslant 0. \tag{3.10}$$

Note that ε_k for $k > 1$ can be expressed in terms of ε_2 and ε_0. For example

$$\varepsilon_2 = \varepsilon_0 - \frac{2\pi N}{r^2}. \tag{3.11}$$

From (3.6)–(3.9) and (3.11) we solve for the partial derivatives and find,

$$\frac{r}{2\pi} \frac{\partial r}{\partial N} = \frac{\varepsilon_0}{\varepsilon_0^2 - \varepsilon_1^2} \tag{3.12}$$

$$\frac{r}{2\pi} \frac{\partial R}{\partial N} = -\frac{\varepsilon_1}{\varepsilon_0^2 - \varepsilon_1^2} \tag{3.13}$$

$$\frac{1}{\pi} \frac{\partial R}{\partial t} = -\frac{\varepsilon_0}{\varepsilon_0^2 - \varepsilon_1^2} \tag{3.14}$$

$$\frac{1}{\pi} \frac{\partial r}{\partial t} = \frac{\varepsilon_1}{\varepsilon_0^2 - \varepsilon_1^2}. \tag{3.15}$$

Note that $\varepsilon_0 \pm \varepsilon_1 \geqslant 0$, since $u''(x) \geqslant 0$ on a set of positive measure and

$$\varepsilon_0 \pm \varepsilon_1 = \int_{-1}^{1} \mathrm{d}s \sqrt{\frac{1 \pm s}{1 \mp s}} u''(rs + R).$$

Now (3.14) and (3.15) imply that

$$\frac{\partial a}{\partial t} = -\frac{\pi}{\varepsilon_0 - \varepsilon_1} < 0 \qquad \frac{\partial b}{\partial t} = -\frac{\pi}{\varepsilon_0 + \varepsilon_1} < 0. \tag{3.16}$$

Therefore, the extreme zeros, a and b [10], of the time-evolved polynomials are strictly decreasing functions of t. As the end points of the spectrum contract with increasing t, the rest of the zeros being squeezed between a and b are expected to be strictly decreasing functions of t.

This result will be shown to hold for all zeros in section 4.

From (3.12)–(3.15) we find the following system of partial-differential equations:

$$\frac{\partial R}{\partial t} = -\frac{r}{2} \frac{\partial r}{\partial N} \qquad \frac{\partial r}{\partial t} = -\frac{r}{2} \frac{\partial R}{\partial N}. \tag{3.17}$$

This system is recognized as the equation of motion for the Toda lattice in the continuum limit [28].

Let $J = (J_{mn}), m, n \geqslant 0$ be the Jacobi matrix with all its entries zero except possibly the ones given by the recurrence coefficients,

$$J_{n,n+1} = 1 \qquad J_{n,n} = \alpha_n(t) \qquad J_{n,n-1} = \beta_n(t). \tag{3.18}$$

Recall that $\beta_n(t) > 0$ for $n > 0$. We now let J evolve in 'time', denoted as t, according to the dynamical equations

$$\frac{\mathrm{d}J}{\mathrm{d}t} = [J, J_+] \tag{3.19}$$

where the J_+ denotes the strictly upper triangular part of J, that is J_+ results from J by replacing all the entries below the main diagonal by zeros [18]. Written in component form the evolution equation is equivalent to the following system of ordinary differential equations

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_n - \beta_{n+1} \qquad n \geqslant 0 \qquad \beta_0 := 0 \tag{3.20}$$

$$\frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_{n-1} - \alpha_n) \qquad n \geqslant 0. \tag{3.21}$$

Note that (3.20) and (3.21) are satisfied by (2.4) and (2.5). In the continuum limit, $\alpha_n(t) \to \alpha(n,t)$, $\beta_n(t) \to \beta(n,t)$, $\alpha_{n-1} - \alpha_n \to -\frac{\partial\alpha}{\partial n}$, $\beta_n - \beta_{n+1} \to -\frac{\partial\beta}{\partial n}$, (3.21) and (3.22) become,

$$\frac{\partial\alpha}{\partial t} = -\frac{\partial\beta}{\partial n} \tag{3.22}$$

$$\frac{\partial\beta}{\partial t} = -\beta\frac{\partial\alpha}{\partial n}. \tag{3.23}$$

With the indentification: $R \leftrightarrow \alpha$ and $r^2/4 \leftrightarrow \beta$, we see that $r(N,t)$ and $R(N,t)$ which solve the functional equations (3.4), (3.5), are the integrals of the continuum Toda equations. Thus r, R and their first partial derivatives with respect to N and t are expressed in terms of integrals involving the derivatives of the 'unevolved' potential, $u(x)$, and therefore in terms of the initial data, $r(N,0)$ and $R(N,0)$. With the condition that $\limsup_{|x|\to\infty} |u(x)/x| = \infty$ the classical moment problem at $t = 0$ has a unique solution [1], this shows that the flow is isospectral, i.e. $J(t) = U^{-1}(t)J(t=0)U(t)$, where $U(t)$ is a unitary transformation. Furthermore the classical moment problem under the one-time flow is also determinate [1]. Consequently, the initial recurrence coefficient $\beta_n(0)(:= \beta_n(0^+))$ satisfies the condition, $\lim_{n\to\infty} \sqrt{\beta_n(0)}/n = 0$.

Let $u(n,t) := \ln[\beta(n,t)]$, and eliminate $\alpha(n,t)$ from the system (3.22) and (3.23), we find a nonlinear wave equation satisfied by $u(n,t)$,

$$\frac{\partial}{\partial n}\left(\mathrm{e}^u \frac{\partial u}{\partial n}\right) = \frac{\partial^2 u}{\partial t^2}. \tag{3.24}$$

In the small amplitude limit, $\mathrm{e}^u \approx 1 + u$, the linear wave equation (with speed of sound equal to unity) is recovered. Equation (3.24) is a second-order quasilinear partial differential equation and can be reduced to its canonical form through the use of characteristics. Using the standard treatment (see for example Garabedian [17]), the pair of partial differential equations can be recast into an equivalent form in which its canonical form can be obtained easily. Renaming the variable n as x, not to be confused with the spectral parameter mentioned earlier, we find

$$\sqrt{\beta}\left(\frac{\partial}{\partial t} + \sqrt{\beta}\frac{\partial}{\partial x}\right)\alpha + \left(\frac{\partial}{\partial t} + \sqrt{\beta}\frac{\partial}{\partial x}\right)\beta = 0 \tag{3.25}$$

$$\sqrt{\beta}\left(\frac{\partial}{\partial t} - \sqrt{\beta}\frac{\partial}{\partial x}\right)\alpha - \left(\frac{\partial}{\partial t} - \sqrt{\beta}\frac{\partial}{\partial x}\right)\beta = 0. \tag{3.26}$$

With the introduction of the characteristics, $\xi(x,t)$ and $\eta(x,t)$ where

$$\frac{\partial}{\partial\xi} := \frac{\partial}{\partial t} + \sqrt{\beta}\frac{\partial}{\partial x} \tag{3.27}$$

$$\frac{\partial}{\partial\eta} := \frac{\partial}{\partial t} - \sqrt{\beta}\frac{\partial}{\partial x} \tag{3.28}$$

the system can be expressed in the canonical form,

$$\frac{\partial\alpha}{\partial\xi} + \frac{1}{\sqrt{\beta}}\frac{\partial\beta}{\partial\xi} = 0 \tag{3.29}$$

$$\frac{\partial\alpha}{\partial\eta} - \frac{1}{\sqrt{\beta}}\frac{\partial\beta}{\partial\eta} = 0. \tag{3.30}$$

An integration gives,

$$\alpha + 2\sqrt{\beta} = f(\eta) \tag{3.31}$$

$$\alpha - 2\sqrt{\beta} = g(\xi) \tag{3.32}$$

and from these,

$$\alpha = \frac{f(\eta) + g(\xi)}{2} \tag{3.33}$$

$$\beta = \frac{[f(\eta) - g(\xi)]^2}{16}. \tag{3.34}$$

We may identify $f(\eta)$ with $b(x,t)$ and $g(\xi)$ with $a(x,t)$. The quantities, a and b are indeed the Riemann invariants. Furthermore, taking a partial derivative of (3.30) with respect to η and a partial derivative of (3.31) with respect to ξ, we find by elimination

$$\frac{\partial^2\alpha}{\partial\xi\partial\eta} = 0 \tag{3.35}$$

$$\frac{\partial^2\sqrt{\beta}}{\partial\xi\partial\eta} = 0 \tag{3.36}$$

two decoupled *linear* partial differential equations for α and β *in* the characteristics variables ξ and η. Using the Riemann invariants instead of ξ and η, we have the Hodograph equations, in which the independent variables are a and b, and t and x are now functions of a and b,

$$\frac{\partial x}{\partial a} - \sqrt{\beta}\frac{\partial t}{\partial a} = 0 \tag{3.37}$$

$$\frac{\partial x}{\partial b} + \sqrt{\beta}\frac{\partial t}{\partial b} = 0. \tag{3.38}$$

Recall that $\sqrt{\beta} = (b-a)/4$, we find by eliminating x, a partial differential equation satisfied by t,

$$\frac{\partial^2 t}{\partial a\partial b} - \frac{1}{2(b-a)}\left(\frac{\partial t}{\partial b} - \frac{\partial t}{\partial a}\right) = 0 \tag{3.39}$$

whose Riemann function is $A(a,b;a_0,b_0)$. The Riemann function depends on the parameters a_0 and b_0 can be found in [17, p 150, exericise 9],

$$A(a,b;a_0,b_0) = \frac{b-a}{\sqrt{(b-a_0)(b_0-a)}}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{(a-a_0)(b_0-b)}{(b-a_0)(b_0-a)}\right). \tag{3.40}$$

Note ${}_2F_1(\frac{1}{2},\frac{1}{2};1;z) = \frac{2}{\pi}\mathrm{K}(\sqrt{z})$, where K is the complete elliptic integral of the first kind. An integral equation for $t(a,b)$ can be derived using Riemann methods (see [17, pp 127–31]);

$$t(R) = A(P;R)t(P) + A(Q;R)t(Q) - A(S;R)t(S) + \int_S^P \mathrm{d}b' \left(-\frac{1}{2(b'-a)} - \frac{\partial A}{\partial b'}\right) t(a,b') + \int_S^Q \mathrm{d}a' \left(-\frac{1}{2(b-a')} - \frac{\partial A}{\partial a'}\right) t(a',b) \tag{3.41}$$

where P, R, Q, and S are the vertices of the rectangle with coordinates (a,b_0), (a_0,b_0), (a_0,b) and (a,b) respectively.

In the next section exact solutions of the discrete and continuum Toda equation that can be obtained by quadrature are given.

4. Explicit solutions of the dynamical equations

We note here that a particular solution of the nonlinear wave equation (3.24) can be obtained by quadrature. With the ansatz,

$$u(n,t) = f(n) + g(t) \tag{4.1}$$

we find

$$\exp(-g(t))\frac{\mathrm{d}^2 g(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}n}\left(\exp(f(n))\frac{\mathrm{d}f(n)}{\mathrm{d}n}\right) = c_1 \tag{4.2}$$

where c_1 is a separation constant. Elementary integration gives,

$$\mathrm{e}^{f(n)} = \frac{c_1}{2}n^2 + c_2 n + c_3. \tag{4.3}$$

Case 1. $c_1 = 0$ In this case

$$g(t) = c_4 t + c_5 \tag{4.4}$$

and

$$\beta(n,t) = (c_2 n + c_3)\exp(c_4 t) \tag{4.5}$$

where the constant c_5 has been absorbed in c_2 and c_3. With β given by (4.4) the relationships (3.22) and (3.33) imply

$$\alpha(n,t) = -c_2 t + c_6 \qquad \text{if } c_4 = 0 \tag{4.6}$$

$$\alpha(n,t) = -c_4 n - \frac{c_2}{c_4}\exp(c_4 t) + c_6 \qquad \text{if } c_4 \neq 0. \tag{4.7}$$

Case 2. $c_1 \neq 0$ In this case

$$\mathrm{e}^{g(t)} = \frac{2c_4}{c_1}\frac{1}{\sinh^2[\sqrt{c}_4[c_5 - t]]} \tag{4.8}$$

if $c_4 \neq 0$. The case $c_4 = 0$ is a limiting case of (4.7). Thus c_1 can be absorbed in c_2, c_3 and c_4 and through renaming the constants we are led to

$$\beta(n,t) = \frac{c_4(n^2 + 2c_2 n + 2c_3)}{\sinh^2[\sqrt{c}_4[c_5 - t]]} \tag{4.9}$$

$$\alpha(n,t) = -\frac{2\sqrt{c_4}}{c_1}[c_1 n + c_2]\coth[\sqrt{c}_4[c_5 - t]] + c_6. \tag{4.10}$$

Thus both $\alpha(n,t)$ and $\beta(n,t)$ diverge at $t = c_5$, provided $c_5 > 0$, but are well-defined for $c_5 < 0$. Note that $c_4 < 0$ is also allowed in (4.9) and (4.10) which results in replacing $\sinh^2$ and coth by $\sin^2$ and cot, respectively. We shall see that both cases can be realized in an explicit model of the Meixner–Pollaczek and Meixner polynomials.

The above analysis points out two possible solutions of the dynamical equations (3.20), (3.21) through the separation of variables in $\beta_n(t)$. Set

$$\beta_n(t) = B(t)B_n. \tag{4.11}$$

Substituting for β_n from (4.11) into (3.21) we obtain

$$\frac{1}{B(t)}\frac{\mathrm{d}B(t)}{\mathrm{d}t} = \alpha_{n-1}(t) - \alpha_n(t)$$

hence $\alpha_n(t) - \alpha_{n-1}(t)$ is independent of n and

$$\alpha_n(t) = C(t) - \frac{n}{B(t)}\frac{\mathrm{d}B(t)}{\mathrm{d}t}. \tag{4.12}$$

Now (4.10) and (3.20) give

$$\frac{1}{B(t)}\left[-\frac{\mathrm{d}C(t)}{\mathrm{d}t} + n\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\ln B(t))\right] = -B_n + B_{n+1}. \tag{4.13}$$

Since B_n is independent of t this indicates that there are constants c_1 and c_2 such that

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} = -c_1 B(t) \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\ln B(t)) = 2c_2 B(t) \tag{4.14}$$

and

$$B_n = c_2 n^2 + c_1 n \tag{4.15}$$

since $B_0 = 0$.

Case 1. $c_2 = 0$ Here we obtain

$$B(t) = c_3\exp(c_4 t) \qquad C(t) = -\frac{c_1 c_3}{c_4}\exp(c_4 t)$$

and

$$\beta_n(t) = c_1 n\exp(c_4 t) \qquad \alpha_n(t) = -\frac{c_1}{c_4}\exp(c_4 t) - c_4 n + c_5 \tag{4.16}$$

where c_3 has been absorbed in c_1.

In the case $c_2 \neq 0$ we integrate the second equation in (4.14) and obtain

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = 2B(t)\sqrt{c_3 + c_2 B(t)}. \tag{4.17}$$

Case 2. $c_2 \neq 0$ *and* $c_3 = 0$ The integration of (4.17) and (4.14) leads to

$$B(t) = \left(c_4 - \sqrt{c_2}t\right)^{-2} \qquad C(t) = c_5 + \frac{c_1}{\sqrt{c_2}}\left(c_4 - \sqrt{c_2}t\right)^{-1}.$$

Thus $B_n = c_2 n^2 + c_1 n$ and we have established

$$\beta_n(t) = \frac{(c_2 n + c_1)n}{(c_4 - \sqrt{c_2}t)^2} \tag{4.18}$$

$$\alpha_n(t) = c_5 - \frac{c_1}{\sqrt{c_2}(c_4 - \sqrt{c_2}t)} - \frac{2n\sqrt{c_2}}{\sqrt{c_2}(c_4 - \sqrt{c_2}t)}. \tag{4.19}$$

Case 3. $c_2c_3 \neq 0$ In the case under consideration (4.17) has the general solution

$$B(t) = \frac{c_3/c_2}{\sinh^2(c_4 \pm \sqrt{c_3}t)}$$

and we see that there is no loss of generality in taking the minus sign in the above solution. Thus

$$B(t) = \frac{c_3/c_2}{\sinh^2(c_4 - \sqrt{c_3}t)} \qquad C(t) = -\frac{c_1\sqrt{c_3}}{c_2}\coth(c_4 - \sqrt{c_3}t). \tag{4.20}$$

Therefore after replacing c_1 by c_1c_2 we obtain

$$\alpha_n(t) = -c_1\sqrt{c_3}\coth(c_4 - \sqrt{c_3}t) - 2n\sqrt{c_3}\coth(c_4 - \sqrt{c_3}t) + c_5 \tag{4.21}$$

$$\beta_n(t) = \frac{nc_3(n + c_1)}{\sinh^2(c_4 - \sqrt{c_3}t)}. \tag{4.22}$$

Observe that $\alpha_n(t)$ and $\beta_n(t)$ in cases 2 and 3 blow up in finite time when $c_4 > 0$ but are finite for all t if $c_4 < 0$. Furthermore the case $c_3 < 0$ and c_4 purely imaginary is allowed since $\beta_n > 0$ and α_n is real. This leads to allowable time bands.

It is interesting to note that all the solutions found through the separation of variables in β and β_n arise from the Meixner, Laguerre, Hermite, or Meixner–Pollaczek polynomials. The Meixner polynomials are orthogonal with respect to the measure [13, section 10.24]

$$w(x; \beta, c) = \sum_{k=0}^{\infty} \frac{c^k(\beta)_k}{k!}\delta(x - k) \qquad 0 < c < 1 \tag{4.23}$$

and has the recurrence coefficients

$$\alpha_n = -n(c + 1) - \beta \qquad \beta_n = cn(n + \beta - 1). \tag{4.24}$$

Thus

$$w(x; \beta, c)\mathrm{e}^{-xt} = w(x; \beta, c\mathrm{e}^{-t})$$

hence the Meixner polynomials can evolve without blowing up. This corresponds to choosing $c_4 < 0$ in the above solutions. On the other hand the Meixner–Pollaczek polynomials [13, section 2.21] have the weight function

$$w^{(\lambda)}(x; \phi) = \frac{(2|\sin\phi|)^{2\lambda-1}}{\pi}\exp(-(\pi - 2\phi)x)|\Gamma(\lambda + \mathrm{i}x)|^2 \tag{4.25}$$

and

$$\alpha_n = (n + \lambda)\cot\varphi \qquad \beta_n = \frac{n(n + 2\lambda - 1)}{4\sin^2\varphi} \qquad 0 < \varphi < \pi, \lambda > 0. \tag{4.26}$$

It is clear that in (4.25) and (4.26) ϕ and $\phi \pm 2j\pi$, $j = 1, 2, \ldots$ lead to the same polynomials. Furthermore

$$w^{(\lambda)}(x; \phi)\mathrm{e}^{-tx} = w^{(\lambda)}(x; \phi - t/2) \tag{4.27}$$

hence at some positive time t the expression $\phi - t/2$ will get outside any interval of the form $(2j\pi, (2j + 1)\pi)$ for j an integer. This causes a blow up in the recursion coefficients. Also note that

$$w^{(\lambda)}(x; \pi + \phi) = w^{(\lambda)}(-x; -\phi). \tag{4.28}$$

This indicates a division of the time domain into a disjoint union of allowable time bands with a blow up at the point separating one time interval from the next.

Finally the Laguerre polynomials

$$\alpha_n = 2n + \alpha + 1 \qquad \beta_n = n(n+\alpha) \qquad \alpha > -1 \tag{4.29}$$

$$w(x;\alpha) = x^{\alpha} e^{-x} \tag{4.30}$$

hence

$$w(x;\alpha) e^{-xt} = w(x+t;\alpha). \tag{4.31}$$

Thus the Laguerre polynomials will evolve without a blow up and fall into case 1.

In the next section certain identities involving the time-evolved zeros are obtained.

5. Time evolution of zeros

In this section we discuss the time evolution of zeros of evolved polynomials as they evolve with time. We employ two methods, Markov's theorem and the Hellmann–Feynman theorem. The version we use of Markov's theorem is more general than the one in Szegö [27] and was stated in [14, chapter III, problem 15] and is elaborated on in [19]. It asserts that if $\{p_n(x,t)\}$ satisfy the orthogonality relation

$$\int_a^b p_m(x,t) p_n(x,t) d\alpha(x,t) = h_n(t)\delta_{m,n} \tag{5.1}$$

and

$$d\alpha(x,t) = \rho(x,t) d\alpha(x) \qquad x \in (a,b) \tag{5.2}$$

then the zeros $X_{N,k}(t)$ of $p_N(x,t)$ satisfy

$$\int_a^b \frac{p_N^2(x,t)}{x - X_{N,k}(t)} \left[\frac{\rho_t(x,t)}{\rho(x,t)} - \frac{\rho_t(X_{N,k}(t),t)}{\rho(X_{N,k}(t),t)}\right] d\alpha(x,t) = A_k(t)\left[p_N'(X_{N,k}(t),t)\right]^2 \frac{\partial X_{N,k}(t)}{\partial t} \tag{5.3}$$

the prime refers to differentiation with respect to x, and $\rho_t := \frac{\partial \rho}{\partial t}$. The numbers A_k are the Christoffel numbers defined through, [27]

$$A_k := \int_a^b \left(\frac{p_N(x;t)}{p_N'(X_{N,k}(t),t)(x - X_{N,k}(t))}\right)^2 d\alpha(x,t) \qquad k = 1, 2, \ldots, N \tag{5.4}$$

where p_N' is, as above, the partial derivative with respect to x. The assumptions under which (5.3) holds are spelled out in the references quoted above. Note that the Christoffel numbers are positive. On the other hand the Hellmann–Feynman theorem asserts that [19]

$$\left[\sum_{j=0}^{N-1} p_j^2(X_{N,k}(t),t)/h_j(t)\right] \frac{\partial X_{N,k}(t)}{\partial t} = \sum_{j=0}^{N-1} p_j(X_{N,k}(t),t) \times \left[p_j(X_{N,k}(t),t)\frac{\partial}{\partial t}\alpha_j(t) + p_{j-1}(X_{N,k}(t),t)\frac{\partial}{\partial t}\beta_j(t)\right] / h_j(t). \tag{5.5}$$

We now come to the question of time evolution of the zeros of $p_N(x,t)$. Following [18] one can study time evolution where time now is a vector $(t_1, t_2, \ldots, t_M)$ with the time-dependent measure being

$$d\alpha(x,t) = \exp\left(-\sum_{j=1}^{M} t_j x^j\right) d\alpha(x) \qquad M < \infty \tag{5.6}$$

where $\alpha(x)$ is independent of the time parameters. In this case the time variable in (2.4), (2.5), (3.22) and (3.23) is t_1. Thus Markov's theorem, (5.3), gives

$$\frac{\partial X_{N,k}(t)}{\partial t_1} = -\frac{h_N(t)/A_k(t)}{[p_N'(X_{N,k}(t),t)]^2}. \tag{5.7}$$

Therefore $X_{N,k}(t)$ strictly decreases with t_1, as one would expect because the additional positive external potential shrinks the ends of the interval. Similarly the dependence of $X_{N,k}(t)$ on t_2 obeys (5.3), which in this case becomes

$$\begin{aligned}\frac{\partial X_{N,k}(t)}{\partial t_2} &= -\frac{1/A_k(t)}{\left[p_N'(X_{N,k}(t),t)\right]^2}\int_{-\infty}^{\infty}[x+X_{N,k}(t)]p_N^2(x,t)\mathrm{d}\alpha(x,t)\\ &= -\frac{[\alpha_N(t)+X_{N,k}(t)]}{\left[p_N'(X_{N,k}(t),t)\right]^2}\frac{h_N(t)}{A_k(t)}.\end{aligned} \tag{5.8}$$

Therefore $X_{N,k}(t)$ will strictly increase or decrease with time according to whether $\alpha_N(t)+X_{N,k}(t)$ is nonpositive or nonnegative.

The Hellmann–Feynman theorem gives different representations for $\frac{\partial X_{N,k}(t)}{\partial t_1}$ and $\frac{\partial X_{N,k}(t)}{\partial t_2}$. For example the dependence of $X_{N,k}(t)$ on t_1 obeys the law

$$\frac{\partial X_{N,k}(t)}{\partial t_1} = \frac{[p_{N-1}(X_{N,k}(t),t)p_{N+1}(X_{N,k}(t),t)]/h_{N-1}(t)}{\sum_{j=0}^{N-1} p_j^2(X_{N,k}(t),t)/h_j(t)}. \tag{5.9}$$

The proof is as follows. Using (4.5) and the Toda equations (3.22) and (3.23) to see that

$$\begin{aligned}\left[\sum_{j=0}^{N-1} p_j^2(X_{N,k}(t),t)/h_j(t)\right]\frac{\partial X_{N,k}(t)}{\partial t_1} &= \sum_{j=0}^{N-1}\frac{(\beta_j-\beta_{j+1})}{h_j(t)}p_j^2(X_{N,k}(t),t)\\ &\quad+\sum_{j=1}^{N-1}\frac{\beta_j(\alpha_{j-1}-\alpha_j)}{h_j(t)}p_j(X_{N,k}(t),t)p_{j-1}(X_{N,k}(t),t).\end{aligned} \tag{5.10}$$

Since $h_n(t)=\prod_{j=1}^n \beta_j(t)$ we can simplify the right-hand side of (5.10) to

$$\begin{aligned}&\sum_{j=0}^{N-2}\frac{p_{j+1}^2(X_{N,k}(t),t)}{h_j(t)} - \sum_{j=0}^{N-1}\frac{\beta_{j+1}p_j^2(X_{N,k}(t),t)}{h_j(t)} + \sum_{j=0}^{N-1}\frac{\alpha_j(t)}{h_j(t)}p_j(X_{N,k}(t),t)p_{j+1}(X_{N,k}(t),t)\\ &\quad-\sum_{j=0}^{N-1}\frac{\alpha_{j+1}(t)}{h_j(t)}p_j(X_{N,k}(t),t)p_{j+1}(X_{N,k}(t),t)\\ &\quad=\sum_{j=0}^{N-2}\frac{p_{j+1}(X_{N,k}(t),t)}{h_j(t)}\left[p_{j+1}(X_{N,k}(t),t)+\alpha_j(t)p_j(X_{N,k}(t),t)\right]\\ &\quad-\sum_{j=0}^{N-1}\frac{p_j(X_{N,k}(t),t)}{h_j(t)}\left[\alpha_{j+1}p_{j+1}(X_{N,k}(t),t)+\beta_{j+1}p_j(X_{N,k}(t),t)\right].\end{aligned}$$

Using the three-term recurrence relation (2.3) we reduce the series on the extreme right-hand side above to a telescoping series and it simplifies and leads to (5.9).

Recall the Christoffel–Darboux formula [27]

$$\sum_{j=0}^{n-1}\frac{p_j^2(x)}{h_j} = \frac{p_{n-1}(x)p_n'(x)-p_{n-1}'(x)p_n(x)}{h_{n-1}} \tag{5.11}$$

valid for monic orthogonal polynomials $\{p_n(x)\}$, whose L^2 norms are $\{\sqrt{h_n}\}$. Let us order the zeros of $p_N(x,t)$ as

$$X_{N,1}(t) > X_{N,2}(t) > \cdots > X_{N,N}(t). \tag{5.12}$$

From (5.11) it follows that $p_{N-1}(x)\,p'_N(x) > 0$ and $p_{N+1}(x)p'_N(x) < 0$ at the zeros of $p_N(x)$. Hence $p_{N-1}(x,t)p_{N+1}(x,t) < 0$ at $x = X_{N,k}(t)$, $1 \leqslant k \leqslant N$. This and (5.9) indicate that $X_{N,k}(t)$ strictly decreases with t. In fact (5.11) gives the following alternate representation for $\frac{\partial X_{N,k}(t)}{\partial t}$

$$\frac{\partial X_{N,k}(t)}{\partial t} = \frac{p_{N+1}(X_{N,k}(t),t)}{p'_N(X_{N,k}(t),t)}. \tag{5.13}$$

It is of interest to compare the left-hand sides of (5.7) and (5.13) because it leads to the curious relationship

$$A_k p_{N+1}(X_{N,k}(t),t)p'_N(X_{N,k}(t),t) + h_N(t) = 0. \tag{5.14}$$

Even more curious identities arise from (5.7) with $k = 1$ and $k = N$ and (3.17) which relate the Christoffel numbers, the derivatives of the polynomials evaluated at the zeros and the ε functions. They are

$$A_N[p'_N(X_{N,N}(t),t)]^2 = (\varepsilon_0 - \varepsilon_1)h_N(t) \tag{5.15}$$

$$A_1[p'_N(X_{N,1}(t),t)]^2 = (\varepsilon_0 + \varepsilon_1)h_N(t). \tag{5.16}$$

One can also use the Hellmann–Feynman theorem to study the t_2 time evolution of the zeros of $p_N(x,t)$ with $\alpha(x,t)$ as in (5.6). The results are now more complicated and we decided not to include them because (5.8) is simple enough and we felt the t_2 results do not add to the understanding of the subject matter. However, in a separate paper we shall give a detail investigation of the special case of the t_2 evolution with an even potential $u(x)$. It can be shown that the recurrence coefficients $\beta(N,t_2)$ satisfies the dispersionless KdV and this will be of interest to the theory of orthogonal polynomials where the spectrum of the Jacobi matrix has gaps. In the next section, we derive using a combination of identities from the theory of orthogonal polynomials and certain Coulomb fluid relations a second-order ordinary differential equation satisfied by the wavefunction, $\varphi_N(x) := \sqrt{w}(x)p_N(x)$, for large N.

6. Differential equations

For the Hermitian matrix ensemble, there are classical examples for which the orthogonal polynomials in addition to satisfying the recurrence relations also satisfy a second-order ordinary differential equation. In order to establish asymptotic properties of polynomials that are orthogonal with respect to nonclassical weight functions, in particular those that are not covered by Bochner's theorem [2], it would be very useful if an *asymptotic* differential equation valid for a large degree could be found. From previous experience, we have conjectured that if the density near its extreme zeros behaves as

$$\sigma(x) \sim G(a,b)\sqrt{b-x} \qquad x \sim b$$

$$\sigma(x) \sim H(a,b)\sqrt{x-a} \qquad x \sim a$$

where

$$G(a,b) = \frac{\sqrt{b-a}}{2\pi^2}\int_a^b \mathrm{d}y\,\frac{v'(b)-v'(y)}{(b-y)^{3/2}\sqrt{y-a}} \tag{6.1}$$

$$H(a,b) = \frac{\sqrt{b-a}}{2\pi^2}\int_a^b \mathrm{d}y\,\frac{v'(y)-v'(a)}{(y-a)^{3/2}\sqrt{b-y}} \tag{6.2}$$

then the *uniform* asymptotics of the polynomials are

$$\sqrt{w(x)}\,p_N(x) \sim \mathrm{Ai}(t_b) \qquad t_b := c_1[G(a,b)]^{2/3}(b-x) \tag{6.3}$$

$$\sqrt{w(x)}\,p_N(x) \sim \mathrm{Ai}(t_a) \qquad t_a := c_2[H(a,b)]^{2/3}(x-a) \tag{6.4}$$

respectively, where Ai(.) is the Airy function, c_1 and c_2 are constants *independent* of N.

With a combination of identities derived by [3], the recurrence relations and the Coulomb fluid method, a *universal* second-order ordinary differential equation is derived and the above conjectures could be proved.

For the sake of completeness the differentiation formula of [3] is reproduced here. From now on we find it convenient to use the orthonormal polynomials $\hat{p}_n := p_n/\sqrt{h_n}$. With these the recurrence relations become,

$$x\hat{p}_n(x,t) = \sqrt{\beta_{n+1}(t)}\,\hat{p}_{n+1}(x,t) + \alpha_n\hat{p}_n(x,t) + \sqrt{\beta_n(t)}\,\hat{p}_{n-1}(x,t). \tag{6.5}$$

To ease notation, the t dependence of the polynomials and the recurrence coefficients is not displayed. Since $\hat{p}_n'(x)$ is a polynomial of degree $n-1$, it can be expanded in terms of $\{\hat{p}_k;\, 0 \leqslant k \leqslant n-1\}$;

$$\hat{p}_n'(x) = \sum_{k=0}^{n-1} c_{kn}\,p_k(x) \tag{6.6}$$

where the expansion coefficients c_{kn} are determined by the orthonormality relation. Thus

$$c_{kn} = \int_{-\infty}^{\infty} \mathrm{d}y\, w(y)\hat{p}_k(y)\hat{p}_n'(y) \qquad 0 \leqslant k \leqslant n-1 \tag{6.7}$$

where again the dependence of the weight function and the potential $v := -\ln w$, on t is not displayed. Using this we find, through integration by parts and noting that $\lim_{x\to\pm\infty} w(x)x^p = 0$,

$$\hat{p}_n'(x) = -\int_{-\infty}^{\infty} \mathrm{d}y\, w(y)[v'(x) - v'(y)]\mathcal{K}_n(x,y)\hat{p}_n(y) \tag{6.8}$$

where

$$\mathcal{K}_n(x,y) := \sum_{k=0}^{n-1} \hat{p}_k(x)\hat{p}_k(y) = \sqrt{\beta_n}\,\frac{\hat{p}_n(x)\hat{p}_{n-1}(y) - \hat{p}_n(y)\hat{p}_{n-1}(x)}{x-y}. \tag{6.9}$$

In the above calculation we have made use of the obvious identity,

$$v'(x)\int_{-\infty}^{\infty} \mathrm{d}y\, w(y)\hat{p}_k(y)\hat{p}_n(y) = 0 \qquad 0 \leqslant k \leqslant n-1 \tag{6.10}$$

to introduce the $v'(x) - v'(y)$ term in (6.8). Thus

$$\hat{p}_n'(x) = -B_n(x)\hat{p}_n(x) + A_n(x)\hat{p}_{n-1}(x) \tag{6.11}$$

with

$$A_n(x) := \sqrt{\beta_n}\int_{-\infty}^{\infty} \mathrm{d}y\, w(y)\frac{v'(x) - v'(y)}{x-y}\hat{p}_n^2(y) \tag{6.12}$$

and

$$B_n(x) := \sqrt{\beta_n}\int_{-\infty}^{\infty} \mathrm{d}y\, w(y)\frac{v'(x) - v'(y)}{x-y}\hat{p}_n(y)\hat{p}_{n-1}(y). \tag{6.13}$$

We now derive an identity between $B_n(x)$ and $A_n(x)$. Using the recurrence relation, we have

$$B_n(x) = \int_{-\infty}^{\infty} \mathrm{d}y\, w(y) \frac{v'(x) - v'(y)}{x - y}\, \hat{p}_n(y)[y\hat{p}_n(y) - \sqrt{\beta}_{n+1}\hat{p}_{n+1}(y) - \alpha_n \hat{p}_n(y)]$$
$$= -B_{n+1}(x) - \frac{\alpha_n}{\sqrt{\beta_n}} A_n(x) + \int_{-\infty}^{\infty} \mathrm{d}y\, w(y) \frac{v'(x) - v'(y)}{x - y}\, y\hat{p}_n^2(y).$$

With the partial fraction decomposition, $\frac{y}{x-y} = \frac{x}{x-y} - 1$, and noting that $w(y)v'(y) = -w'(y)$, followed by an integration by parts and discarding boundary terms, gives

$$B_n(x) + B_{n+1}(x) = \frac{x - \alpha_n}{\sqrt{\beta_n}} A_n(x) - v'(x). \tag{6.14}$$

We now derive the differential equation satisfied by the wavefunction,

$$\varphi_N(x) := \exp[-v(x)/2]\hat{p}_N(x)$$

for sufficiently large N. First note that very large N, $B_N(x)$ is a slowly varying function of N, and $B_N(x) \approx B_{N+1}(x)$. This gives

$$B_N(x) = \frac{x - \alpha_N}{2\sqrt{\beta_N}} A_N(x) - \frac{v'(x)}{2}. \tag{6.15}$$

We also note that as α_N and β_N are thermodynamics quantities $\alpha_{N-1} \approx \alpha_N \approx_{N+1}$ and $\beta_{N-1} \approx \beta_N \approx \beta_{N+1}$. Also $A_{N-1}(x) \approx A_N(x) \approx A_{N+1}(x)$, this will become clear later. Substituting $\hat{p}_N(x) = \exp[v(x)/2]\varphi_N(x)$ into the differentiation formula (6.11) and make use of (6.15) we find

$$\varphi_N'(x) = A_N(x)\left[\varphi_{N-1}(x) - \frac{x - \alpha_N}{2\sqrt{\beta_N}}\varphi_N(x)\right]. \tag{6.16}$$

Now in the same approximation, the recurrence relation becomes,

$$\frac{x - \alpha_N}{\sqrt{\beta_N}}\varphi_N(x) = \varphi_{N+1}(x) + \varphi_{N-1}(x). \tag{6.17}$$

Using the large N recurrence relations, (6.17), we find by eliminating $\frac{x-\alpha_N}{\sqrt{\beta_N}}\varphi_N(x)$, an alternative expression for (6.16),

$$\varphi_N'(x) = \frac{A_N(x)}{2}\left[\varphi_{N+1}(x) - \varphi_{N-1}(x)\right]. \tag{6.18}$$

Note that we have taken care *not* to approximate $\varphi_{N+1}(x)$ and $\varphi_{N-1}(x)$ by $\varphi_N(x)$, as the wavefunction varies rapidly with N. Differentiating (6.18) with respect to x, we find,

$$\varphi_N''(x) = \frac{A_N'(x)}{2}\left[\varphi_{N-1}(x) - \varphi_{N+1}(x)\right] + \frac{A_N(x)}{2}\left[\varphi_{N-1}'(x) - \varphi_{N+1}'(x)\right]$$
$$= (\ln A_N(x))'\,\varphi_N'(x) + \frac{A_N(x)}{2}\left[\varphi_{N-1}'(x) - \varphi_{N+1}'(x)\right]. \tag{6.19}$$

Now using the (6.17) again on (6.18), this time eliminating $\varphi_{N-1}(x)$, we find a second differentiation formula,

$$\varphi_N'(x) = A_N(x)\left[\frac{x - \alpha_N}{2\sqrt{\beta_N}}\varphi_N(x) - \varphi_{N+1}(x)\right]. \tag{6.20}$$

Replace N by $N+1$ in (6.16) and N by $N-1$ in (6.20), and a subtraction gives,

$$\begin{aligned}\varphi'_{N-1}(x)-\varphi'_{N+1}(x)&=A_N(x)\left[\frac{x-\alpha_N}{2\sqrt{\beta}}(\varphi_{N+1}(x)+\varphi_{N-1}(x))-2\varphi_N(x)\right]\\&=A_N(x)\left[\frac{1}{2}\left[\frac{x-\alpha_N}{\sqrt{\beta}_N}\right]^2-2\right]\varphi_N(x)\end{aligned}\tag{6.21}$$

where we have used the large N recurrence relations and the approximation on the recurrence coefficients mentioned. In this way $\varphi'_{N+1}(x)$ and $\varphi'_{N-1}(x)$ are eliminated and we have the differential equation,

$$\varphi''_N(x)-(\ln A_N(x))'\varphi'_N+A_N^2(x)\left[1-\left(\frac{x-\alpha_N}{2\sqrt{\beta}_N}\right)^2\right]\varphi_N(x)=0.\tag{6.22}$$

As it stands (6.22) is quite useless as we do not have any information on $A_N(x)$ that are expressed in terms of the polynomials that we seek in the first place. In the Coulomb fluid method, as shown in section 2, $\sqrt{\beta}_N=\frac{b-a}{4}$ and $\alpha_N=\frac{b+a}{2}$ where a and b are edge parameters that determine the termination points of the fluid density. Using these, a simple calculation shows that

$$\varphi''_N(x)-(\ln A_N(x))'\varphi'_N(x)+\frac{A_N^2(x)}{4\beta_N}(x-a)(b-x)\varphi_N(x)=0.\tag{6.23}$$

This suggests that there is a simple relation between $A_N(x)$ and fluid density $\sigma(x)$. Indeed this is the case. From the expression of $A_k(x)$, we introduced,

$$C_N(x):=\sum_{k=0}^{N-1}\frac{A_k(x)}{\sqrt{\beta}_k}=\int_{-\infty}^{\infty}\mathrm{d}y\,\frac{v'(x)-v'(y)}{x-y}\sigma_N(y)\tag{6.24}$$

where

$$\sigma_N(x):=w(x)\mathcal{K}_N(x,x)\tag{6.25}$$

is the *exact* zero-counting function or the *exact* density, which can be approximated in the Coulomb fluid method by the continuum density satisfied by the integral equation. A telescopic sum gives,

$$\begin{aligned}\frac{A_N(x)}{\sqrt{\beta}_N}&=C_{N+1}(x)-C_N(x)=\int_{-\infty}^{\infty}\mathrm{d}y\,\frac{v'(x)-v'(y)}{x-y}\left[\sigma_{N+1}(y)-\sigma_N(y)\right]\\&\approx\int_{a(N+1,t)}^{b(N+1,t)}\mathrm{d}y\,\frac{v'(x)-v'(y)}{x-y}\sigma(y;N+1)-\int_{a(N,t)}^{b(N,t)}\frac{v'(x)-v'(y)}{x-y}\sigma(y,N)\\&=\int_a^b\mathrm{d}y\,\frac{v'(x)-v'(y)}{x-y}\frac{\partial\sigma(y)}{\partial N}+\mathrm{o}(1)\\&=\int_a^b\mathrm{d}y\,\frac{v'(x)-v'(y)}{x-y}\frac{1}{\pi\sqrt{(b-y)(y-a)}}\\&=\frac{P}{\pi}\int_a^b\frac{\mathrm{d}y\,v'(y)}{(y-x)\sqrt{(b-y)(y-a)}}\\&=\frac{2\pi\sigma(x)}{\sqrt{(b-x)(x-a)}}=\pi^2\frac{\partial\sigma^2(x)}{\partial N}.\end{aligned}\tag{6.26}$$

Thus

$$\left[\frac{A_N(x)}{2\sqrt{\beta}_N}\sqrt{(b-x)(x-a)}\right]^2=\pi^2\sigma^2(x).\tag{6.27}$$

We also note that

$$A_N(x) = \pi^2\sqrt{\beta}_N \frac{\partial\sigma^2(x)}{\partial N} \tag{6.28}$$

implies

$$(\ln A_N(x))' = \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{\partial\sigma^2(x)}{\partial N}\right). \tag{6.29}$$

The differential equation reads

$$\varphi_N''(x) - \left[\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{\partial\sigma^2(x)}{\partial N}\right)\right]\varphi_N'(x) + \pi^2\sigma^2(x)\varphi_N(x) = 0. \tag{6.30}$$

This is the asymptotic differential equation that we seek with coefficients expressed in terms of a known quantity, the density $\sigma(x)$.

From (6.27) or (6.28), we can determine the behaviour of $A_N(x)$ near the edges of the spectrum of the Jacobi matrix, a and b. A simple calculation shows that,

$$\begin{aligned}
&A_N(x) \sim \sqrt{\beta}_N \frac{2\pi G(a,b)}{\sqrt{b-a}} && x \sim b\\
&\sigma^2(x) \sim G^2(a,b)(b-x) && x \sim b\\
&A_N(x) \sim \sqrt{\beta}_N \frac{2\pi H(a,b)}{\sqrt{b-a}} && x \sim a\\
&\sigma^2(x) \sim H^2(a,b)(x-a) && x \sim a
\end{aligned}$$

and the coefficients of $\varphi_N'(x)$ vanish. With the variables

$$\begin{aligned}
t_b &:= [\pi G(a,b)]^{2/3}(x-b)\\
t_a &:= [\pi H(a,b)]^{2/3}(a-x)
\end{aligned} \tag{6.31}$$

we see that the wavefunction satisfies the Airy equation,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\lambda^2} - \lambda y = 0 \tag{6.32}$$

where λ is t_b or t_a depending on the appropriate edges that we scale to. This establishes the conjecture stated. Further insight into the polynomials or the wavefunction in the semi-classical limit can be gained by examining the differentiation formulae. With

$$g(x) := \exp\left[\int^x A_N(s)\left[\frac{s-\alpha_N}{2\sqrt{\beta}_N}\right]\mathrm{d}s\right] \tag{6.33}$$

the differentiation formulae becomes,

$$(\varphi_N(x)g(x))' = A_N(x)\varphi_{N-1}(x)g(x) \tag{6.34}$$

$$\left(\frac{\varphi_{N-1}(x)}{g(x)}\right)' = -A_N(x)\frac{\varphi_N(x)}{g(x)}. \tag{6.35}$$

Eliminating $\varphi_{N-1}(x)$, gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{A_N(x)g^2(x)}\frac{\mathrm{d}}{\mathrm{d}x}(\varphi_N(x)g(x))\right] = -\frac{A_N(x)}{g^2(x)}(\varphi_N(x)g(x)). \tag{6.36}$$

Introducing the new spectral variable, ξ, through,

$$\mathrm{d}\xi := A_N(x)g^2(x)\mathrm{d}x \tag{6.37}$$

and $Y_N(\xi) := \varphi_N(x)g(x)$, the differential equation becomes,

$$\frac{\mathrm{d}^2 Y_N}{\mathrm{d}\xi^2} + \frac{Y_N(\xi)}{g^4} = 0. \tag{6.38}$$

Using the WKB method, we find that the semiclassical wavefunction is given as a suitable linear combination of

$$\varphi_{\pm}(x) \sim \exp\left[\pm\mathrm{i}\int^x \mathrm{d}s\, A_N(s)\right]. \tag{6.39}$$

For a class of orthogonal polynomials that arises from the theory of quantum transport in disordered systems [7, 4], the weight function decreases slowly so that the potential, $u(x)$, confines the charges only weakly, i.e. $\lim_{|x|\to\infty} u(x)/x = 0$, for $x \in \mathbb{R}$, in this case the moment problem is indeterminate and it can be shown that [1], $\lim_{N\to\infty}\sigma_N(x)$ exists for fixed x. Therefore, the density $\sigma(x)$ has a $N \to \infty$ limit, denoted by $\varrho(x)$. From (6.26) and assuming $u(x)$ is even (which implies $\alpha_N = 0$), it is clear that $\lim_{N\to\infty} A_N(x) = \pi\varrho(x)$, and $\lim_{N\to\infty} A_N(x)/\sqrt{\beta_N} = 0$. Denoting $\lim_{N\to\infty}[\beta_N]^{1/4}\varphi_N(x) = e_1(x)$ and $\lim_{N\to\infty}[\beta_N]^{1/4}\varphi_{N-1}(x) = e_2(x)$, we obtain the following differentiation formulae for $e_1(x)$ and $e_2(x)$. The differentiation formulae (6.16) and (6.20) (replacing N by $N-1$), in the limit $N \to \infty$, become

$$e_1'(x) = \pi\varrho(x)e_2(x) \tag{6.40}$$

$$e_2'(x) = -\pi\varrho(x)e_1(x). \tag{6.41}$$

Thus, $e_1(x)$ and $e_2(x)$ satisfy the differential equation

$$y''(x) - [\ln\varrho(x)]'y(x) + \pi^2\varrho^2(x)y(x) = 0 \tag{6.42}$$

and we have from these the reproducing kernel conjectured in [8],

$$\lim_{N\to\infty} K_N(x,y) := \lim_{N\to\infty}\sqrt{w(x)}\mathcal{K}_N(x,y)\sqrt{w(y)} = \frac{e_1(x)e_2(y) - e_2(x)e_1(y)}{x-y}. \tag{6.43}$$

Using the WKB approximation on (6.42), the function $e_j(x)$, $j = 1, 2$, are linear combinations of

$$y_{\pm}(x) \sim \frac{1}{\sqrt{\pi}}\exp\left[\pm\mathrm{i}\pi\int^x \varrho(s)\,\mathrm{d}s\right]. \tag{6.44}$$

This result was conjectured in [8].

7. Conclusion and summary

We have obtained the large N recurrence coefficients of polynomials orthogonal with respect to a class of weight function supported in $\boldsymbol{R}$ as thermodynamics susceptibilities of a Coulomb fluid in one dimension. These polynomials arise naturally in the theory of large Hermitian random matrices [24]. The polynomials are shown to satisfy a second-order ordinary differential equation, thus generalizing the theorem of Bochner. These are of particular interest since the reproducing kernel $K_N(x,y)$ plays an important role in the computation of the variance of linear statistics and the level spacing distribution in random matrix theory [24]. Using the differential equation, we proved a conjecture on the semiclassical behaviour of the polynomials arising from a class of indeterminate moment problems [8] and a conjecture on the 'edge' asymptotic of the orthogonal polynomials stated in [9]

The situation where the spectrum of the Jacobi matrix has gaps will be presented in a separate paper along with the the associated differential equation.

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